

Theta and zeta functions for odd-dimensional locally symmetric spaces of rank one

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February 7, 2008

Contents

1	Introduction	0
2	The trace formula	2
2.1	The restriction map $r : R(\mathbf{K}) \rightarrow R(\mathbf{M})$	2
2.2	The contribution of the identity	5
2.3	The hyperbolic contribution	8
2.4	The distributional trace formula	11
3	Theta functions	12
4	Zeta functions	15
4.1	The logarithmic derivative	15
4.2	The fundamental properties of the zeta functions	18
5	The Ruelle zeta function	21
5.1	Definition and relation with the Selberg zeta function	21
5.2	The functional equation for the Ruelle zeta function	22
5.3	Analytic torsion and the Ruelle zeta function	24

1 Introduction

This paper is a continuation of our work on theta and zeta functions [3], [4] and [5]. In the previous papers we considered the case of even dimensional rank one symmetric spaces

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of non-compact type. The present is concerned with the odd-dimensional case, i.e. with odd-dimensional real hyperbolic manifolds.

It is the natural appearance of eta invariants in connection with super theta and zeta functions what makes this case particularly interesting. That the eta invariant is connected with special value of a zeta function was first observed for the signature operator by Millson [14] and later by Moscovici/Stanton [15] in the general case (even for higher rank spaces). The super theta function was introduced by Juhl [12]. He derived its properties using the super zeta function while the latter is studied through dynamical Lefschetz formulas and by specializing the results of [15].

In odd dimensions the analytic torsion (introduced by Ray/Singer [17]) is another interesting spectral invariant. It is associated to a finite dimensional unitary representation of the fundamental group yielding a flat vector bundle over the locally symmetric space that is used to twist the Laplacians on differential forms. The analytic torsion is then given by a certain combination of zeta-regularized determinants of these twisted Laplacians. It was first observed by Fried [8] that the value of the twisted Ruelle zeta function at zero is the inverse of the analytic torsion. The connection of special values of zeta functions with analytic torsion and (higher) analytic torsion was generalized to higher rank cases by Moscovici/Stanton [16] and Deitmar [6].

Thus, the odd-dimensional case is in some sense exhausted. Moreover, in this case the identity contribution to the Selberg trace formula is much simpler than in the even dimensional case since the Plancherel densities are polynomials. In other terms, the distributional local trace of the corresponding wave operators is a finite sum of derivatives of delta distributions concentrated at zero (see Corollary 2.5), i.e. a weak sort of Huygens' principle holds. This a priori excludes the difficulties with normalization constants one was confronted with in the even dimensional case and which were a major motivation for us to write [3],[4]. There we found an easy solution using the connection of the analysis on the symmetric spaces and its compact dual space. Though not as essential as in the even dimensional case we apply this duality in the present paper in order to give an easy derivation of the identity contribution to the Selberg trace formula avoiding harmonic analysis on the non-compact symmetric space.

Our main motivation to write this paper was to show that one can recover all previous results on theta and zeta functions by a "unique continuation" of our methods of [4]. We believe that our results, even if not explicitly stated in the literature, are known to the specialists in the field. But we think that our approach is rather short and much less involved than the previous ones.

As in [4] our plan is as follows. Let $\mathbf{G} = SO(n, 1)$ or $\mathbf{G} = Spin(n, 1)$ and $\mathbf{K} = SO(n)$ or $\mathbf{K} = Spin(n)$ be a maximal compact subgroup. The zeta and theta functions are associated to \mathbf{M} -types, \mathbf{M} being defined as the centralizer of an Iwasawa \mathbf{a} . We first obtain \mathbf{K} -types restricting to a given \mathbf{M} -type. We use this \mathbf{K} -types in order to define associated vector bundles over the locally symmetric space and certain differential operators. We then prove trace formulas for functions of these operators. Once having the trace formula we study the theta, the Selberg zeta and the Ruelle zeta function in a way similar to [4]. Because of the application to analytic torsion we include twists, i.e. finite dimensional

unitary representations of the fundamental group of the locally symmetric space. Like in the present paper it is easy to include twists in [4] as well.

2 The trace formula

2.1 The restriction map $r : R(\mathbf{K}) \rightarrow R(\mathbf{M})$

We consider the pairs of groups (\mathbf{K}, \mathbf{M}) with $\mathbf{K} = Spin(n)$ and $\mathbf{M} = Spin(n-1)$ or $\mathbf{K} = SO(n)$ and $\mathbf{M} = SO(n-1)$ for an odd integer $n \geq 3$. Let $R(\mathbf{K})$ and $R(\mathbf{M})$ be the corresponding representation rings over \mathbf{Z} . In this subsection we shall investigate the restriction map

$$r : R(\mathbf{K}) \longrightarrow R(\mathbf{M})$$

induced by the inclusion $\mathbf{M} \hookrightarrow \mathbf{K}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g} = so(1, n)$ such that $\mathbf{M} \subset \mathbf{K}$ can be identified with the centralizer of \mathfrak{a} . Let $W := W(\mathfrak{g}, \mathfrak{a})$ be the Weyl group of the restricted root system and $w \in W$ its non-trivial element. w acts on $\hat{\mathbf{M}}$ by

$$w\sigma(m) := \sigma(m_w^{-1}mm_w) \quad m \in \mathbf{M}, \sigma \in \hat{\mathbf{M}},$$

where m_w is a representative of w belonging to the normalizer of \mathfrak{a} in \mathbf{K} . This action extends to $R(\mathbf{M})$. Then $R(\mathbf{M})$ splits (over $\mathbf{Z}[\frac{1}{2}]$) into the ± 1 -eigenspaces of w

$$R(\mathbf{M}) \otimes \mathbf{Z}[\frac{1}{2}] = R(\mathbf{M})^+ \otimes \mathbf{Z}[\frac{1}{2}] \oplus R(\mathbf{M})^- \otimes \mathbf{Z}[\frac{1}{2}].$$

We consider the Cartan subalgebra \mathfrak{t} of $\mathfrak{m} = so(n-1)$, which can also be considered via the inclusion $\mathfrak{m} \hookrightarrow \mathfrak{k}$ as a Cartan subalgebra of $\mathfrak{k} = so(n)$, given by

$$\mathfrak{t} := \left\{ T_\mu := \begin{pmatrix} 0 & -\mu_1 & & \\ \mu_1 & 0 & & \\ & & \ddots & \\ & & & 0 & -\mu_k \\ & & & \mu_k & 0 \end{pmatrix} \mid \mu_i \in \mathbf{R}, k = \frac{n-1}{2} \right\}.$$

We denote by $\nu = (\nu_1, \dots, \nu_k)$ the element in $\imath\mathfrak{t}^*$, which sends T_μ to $\imath(\nu_1\mu_1 + \dots + \nu_k\mu_k)$. We choose systems of positive roots

$$\Phi^+(\mathbf{k}^c, \mathfrak{t}) := \{e_i \pm e_j (i < j), e_i\} \text{ and } \Phi^+(\mathbf{m}^c, \mathfrak{t}) := \{e_i \pm e_j (i < j)\},$$

where $e_i := (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the i 'th entry. Their Weyl groups are given by

$$\begin{aligned} W_k &= \{\text{permutations of the coordinates with possible sign changes}\}, \\ W_m &= \{\text{permutations of the coordinates with an even number of sign changes}\}, \end{aligned}$$

and we have

$$\begin{aligned}\rho_k &:= \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathbf{k}^c, \mathbf{t})} \alpha = (k - \frac{1}{2}, k - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}), \\ \rho_m &:= \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathbf{m}^c, \mathbf{t})} \alpha = (k - 1, k - 2, \dots, 1, 0).\end{aligned}$$

Now by the theory of the highest weights we can write

$$\hat{\mathbf{K}} = \{\gamma_\nu \mid \nu_1 \geq \dots \geq \nu_k \geq 0, \nu_i \in \mathbf{Z}, i = 1, \dots, k \text{ or, if } \mathbf{K} = Spin(n), \nu_i \in \frac{1}{2}\mathbf{Z}\}$$

and

$$\hat{\mathbf{M}} = \{\sigma_\nu \mid \nu_1 \geq \dots \geq \nu_{k-1} \geq |\nu_k|, \nu_i \in \mathbf{Z}, i = 1, \dots, k \text{ or, if } \mathbf{M} = Spin(n-1), \nu_i \in \frac{1}{2}\mathbf{Z}\}.$$

Then the action of w on $\hat{\mathbf{M}}$ looks as follows : $w\sigma_{(\nu_1, \dots, \nu_k)} = \sigma_{(\nu_1, \dots, -\nu_k)}$. In particular, for the half-spin representations $s^\pm = \sigma_{(1/2, \dots, \pm 1/2)}$ of $Spin(n-1)$ we have $ws^\pm = s^\mp$. We denote by s the spin representation of $Spin(n)$. The first part of the following proposition can already be found in Miatello/Vargas [13].

Proposition 2.1 *The map r has the following properties :*

1. r is a bijection between $R(\mathbf{K})$ and $R(\mathbf{M})^+$.
2. If $\sigma \in R(\mathbf{M})^-$, then there exists a unique element $\gamma \in R(Spin(n))$ with $\sigma = (s^+ - s^-)r(\gamma)$ such that $s \cdot \gamma \in R(\mathbf{K})$.
3. More explicitly, if $\nu_k > 0$, then

$$\sigma_{(\nu_1, \dots, \nu_k)} - w\sigma_{(\nu_1, \dots, \nu_k)} = (s^+ - s^-)r(\gamma_{(\nu_1 - \frac{1}{2}, \dots, \nu_k - \frac{1}{2})}). \quad (1)$$

In addition, the \mathbf{K} -representation $s \otimes \gamma_{\nu - (\frac{1}{2}, \dots, \frac{1}{2})}$ splits into two representations $\gamma^+(\nu)$ and $\gamma^-(\nu)$ such that

$$\sigma_{(\nu_1, \dots, \nu_k)} + w\sigma_{(\nu_1, \dots, \nu_k)} = r(\gamma^+(\nu) - \gamma^-(\nu)). \quad (2)$$

Proof: We employ the Weyl character formula. Let us denote by χ_σ (χ_γ) the character of an irreducible representation σ of $Spin(n-1)$ (γ of $Spin(n)$). We introduce the Weyl denominators

$$\begin{aligned}\Delta_{\mathbf{K}}(\exp T) &:= \prod_{\alpha \in \Phi^+(\mathbf{k}^c, \mathbf{t})} (e^{\alpha(T)/2} - e^{-\alpha(T)/2}), \quad T \in \mathbf{t}, \\ \Delta_{\mathbf{M}}(\exp T) &:= \prod_{\alpha \in \Phi^+(\mathbf{m}^c, \mathbf{t})} (e^{\alpha(T)/2} - e^{-\alpha(T)/2}).\end{aligned}$$

Note that $\chi_{s^+}(expT) - \chi_{s^-}(expT) = \Delta_{\mathbf{K}}(expT)/\Delta_{\mathbf{M}}(expT)$. For a highest weight ν of $Spin(n)$ we compute

$$\begin{aligned}
& ((\chi_{s^+} - \chi_{s^-})\chi_{\gamma_\nu})(expT) \\
&= (\chi_{s^+} - \chi_{s^-})(expT) \frac{\sum_{s \in W_k} detse^{s(\nu+\rho_k)(T)}}{\Delta_{\mathbf{K}}(expT)} \\
&= \frac{\sum_{s \in W_k} detse^{s(\nu+\rho_k)(T)}}{\Delta_{\mathbf{M}}(expT)} \\
&= \frac{\sum_{s \in W_k} detse^{s(\nu+(1/2, \dots, 1/2)+\rho_m)(T)}}{\Delta_{\mathbf{M}}(expT)} \\
&= \frac{\sum_{s \in W_m} detse^{s(\nu+(1/2, \dots, 1/2)+\rho_m)(T)} - e^{s(\nu+(1/2, \dots, 1/2, -2\nu_k-1/2)+\rho_m)(T)}}{\Delta_{\mathbf{M}}(expT)} \\
&= \chi_{\sigma_{\nu+(1/2, \dots, 1/2)}}(expT) - \chi_{w\sigma_{\nu+(1/2, \dots, 1/2)}}(expT) .
\end{aligned} \tag{3}$$

Thus we have proved (1), and the second part of the proposition follows. Further we conclude that r is injective, since in view of (3) the map $R(K) \ni \gamma \rightarrow (s^+ - s^-)r(\gamma) \in R(Spin(n-1))$ has no kernel. In fact

$$\sigma_{\nu+(1/2, \dots, 1/2)} \neq w\sigma_{\nu+(1/2, \dots, 1/2)}.$$

Since the action of w interchanges the \mathbf{M} -isotypic components of an element in $\hat{\mathbf{K}}$, we see that the image of r is contained in $R(\mathbf{M})^+$. In order to prove the first assertion of the proposition it remains to construct a preimage in $R(\mathbf{K})$ for every $\sigma \in R(\mathbf{M})^+$. Since $(s^+ - s^-)\sigma \in R(Spin(n-1))^-$, we find an element $\gamma \in R(Spin(n))$, which actually belongs to $R(\mathbf{K})$, such that

$$(s^+ - s^-)\sigma = (s^+ - s^-)r(\gamma) .$$

Now $R(Spin(n-1))$ is a polynomial ring, hence $\sigma = r(\gamma)$.

We are left with the proof of (2). Let us consider the character of $s \otimes \gamma_\nu$

$$\begin{aligned}
\chi_s \chi_{\gamma_\nu}(expT) &= \sum_{v \in \{\pm 1\}^k} \sum_{s \in W_k} \frac{detse^{(s(\nu+\rho_k)+v(\frac{1}{2}, \dots, \frac{1}{2}))(T)}}{\Delta_{\mathbf{K}}(expT)} \\
&= \sum_{v \in \{\pm 1\}^k} \sum_{s \in W_k} \frac{detse^{s(\nu+v(\frac{1}{2}, \dots, \frac{1}{2})+\rho_k)(T)}}{\Delta_{\mathbf{K}}(expT)} \\
&= \sum_{\{v \in \{\pm 1\}^k \mid \nu+v(\frac{1}{2}, \dots, \frac{1}{2})+\rho_k \text{ regular in } \mathfrak{t}^*\}} \sum_{s \in W_k} \frac{detse^{s(\nu+v(\frac{1}{2}, \dots, \frac{1}{2})+\rho_k)(T)}}{\Delta_{\mathbf{K}}(expT)} .
\end{aligned} \tag{4}$$

In the last step we have used that for singular ξ

$$\sum_{s \in W_k} detse^{s\xi(T)} \equiv 0 .$$

Observe that $\nu + v(\frac{1}{2}, \dots, \frac{1}{2})$ is a highest weight of an irreducible representation of \mathbf{K} iff $\nu + v(\frac{1}{2}, \dots, \frac{1}{2}) + \rho_k$ is regular. Therefore we can define

$$\gamma^\pm(\nu) := \sum_{\{v \in \{\pm 1\}^k \mid \nu + v(\frac{1}{2}, \dots, \frac{1}{2}) + \rho_k \text{ regular in } \mathfrak{t}^*, \det v = \pm 1\}} \gamma_{\nu + v(\frac{1}{2}, \dots, \frac{1}{2})},$$

and (4) says that $s \otimes \gamma_\nu = \gamma^+(\nu) \oplus \gamma^-(\nu)$.

For $\xi = (\xi_1, \dots, \xi_k)$ we set $\xi^- := (\xi_1, \dots, \xi_{k-1}, -\xi_k)$. Let p be the projection $p : W_k \rightarrow \{\pm 1\}^k$ given by $p(s)((\frac{1}{2}, \dots, \frac{1}{2})) := s((\frac{1}{2}, \dots, \frac{1}{2}))$. Then we have

$$\begin{aligned} & (\Delta_{\mathbf{K}}(\chi_{\sigma_{\nu+(1/2, \dots, 1/2)}} + \chi_{w\sigma_{\nu+(1/2, \dots, 1/2)}}))(expT) \\ &= (\Delta_{\mathbf{M}}(\chi_{s^+} - \chi_{s^-})(\chi_{\sigma_{\nu+(1/2, \dots, 1/2)}} + \chi_{w\sigma_{\nu+(1/2, \dots, 1/2)}}))(expT) \\ &= \left(\sum_{v \in \{\pm 1\}^k} \det v e^{v(\frac{1}{2}, \dots, \frac{1}{2})(T)} \right) \left(\sum_{s \in W_m} \det s (e^{s(\nu + \rho_k)(T)} + e^{s((\nu + \rho_k)^-)(T)}) \right) \\ &= \left(\sum_{v \in \{\pm 1\}^k} \det v e^{v(\frac{1}{2}, \dots, \frac{1}{2})(T)} \right) \left(\sum_{s \in W_k} \det s \det p(s) e^{s(\nu + \rho_k)(T)} \right) \\ &= \sum_{v \in \{\pm 1\}^k} \sum_{s \in W_k} \det v \det s e^{s(\nu + v(\frac{1}{2}, \dots, \frac{1}{2}) + \rho_k)(T)} \\ &= (\Delta_{\mathbf{K}}(\chi_{\gamma^+(\nu)} - \chi_{\gamma^-(\nu)}))(expT). \end{aligned}$$

Now (2) follows. \square

2.2 The contribution of the identity

Let $\sigma \in \hat{\mathbf{M}}$. We have to distinguish the two cases (a): $\sigma = w\sigma$ and (b): $\sigma \neq w\sigma$, where w is the non-trivial element of the Weyl group of (\mathbf{g}, \mathbf{a}) . In the following we apply Proposition 2.1 several times. Let $\gamma \in R(\mathbf{K})$ be the unique lift of σ , i.e. $r(\gamma) = \sigma$ in the case (a) and $r(\gamma) = \sigma + w\sigma$ in the case (b). In the case (b) we also consider the super lift $\gamma^s = s \otimes \gamma' \in R(\mathbf{K})$. s is the spinor representation of $Spin(n)$ and $\gamma' \in Spin(n)^\wedge$ is irreducible such that $\sigma - w\sigma = \pm(s^+ - s^-) \otimes r(\gamma')$ holds in $R(Spin(n-1))$, where s^\pm are the half spinor representations of $Spin(n-1)$ and $r : R(\mathbf{K}) \rightarrow R(\mathbf{M})$ (or $r : R(Spin(n)) \rightarrow R(Spin(n))$) is the restriction homomorphism. If $\mathbf{K}, \mathbf{M} \neq Spin(n), Spin(n-1)$ we view $R(\mathbf{K}), R(\mathbf{M})$ as subrings of $R(Spin(n)), R(Spin(n-1))$. Let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the Cartan decomposition of \mathbf{g} . Then there is a Clifford multiplication $\mathbf{p} \otimes s \rightarrow s$. In fact, there are two Clifford multiplications differing by a sign. Let $\mathbf{a} \subset \mathbf{p}$ be the one-dimensional subspace and $H \in \mathbf{a}$ be the unit vector pointing into the positive direction of \mathbf{a} given by the Iwasawa decomposition. Then we fix the sign of the Clifford multiplication (and hence of the Dirac operator later on) by requiring that H acts as multiplication by $\pm i$ on $s^\pm \subset s$. Since $\mathbf{M} \subset \mathbf{K}$ is defined as the centralizer of \mathbf{a} and thus H is \mathbf{M} -invariant the subspaces $s^\pm \subset s$ are well defined and H respects the decomposition $s = s^+ \oplus s^-$. Note that by the third part of Proposition 2.1 we can choose the representative of the lift γ of $\sigma + w\sigma$ such that $|\gamma| = |\gamma^s|$, i.e. both representations coincide up to the Z_2 -grading. Here

we abuse the notation denoting the element of $R(\mathbf{K})$ and its representative, i.e. a formal sum of irreducible representations with integer coefficients by the same symbol γ . $|\gamma|$ is only defined for the representative.

Let $V^{(s)} = V(\gamma^{(s)})$ be the associated bundle over $X := \mathbf{G}/\mathbf{K}$. It is a \mathbf{G} -homogeneous bundle. Let $-\Omega$ be the Casimir operator of \mathbf{G} acting on sections of $V^{(s)}$ and $\Omega_{\mathbf{M}}$ be the Casimir operator of \mathbf{M} . We normalize the invariant scalar product on \mathfrak{g} such that it induces a metric of sectional curvature -1 on X . The invariant scalar product of \mathfrak{m} is obtained by restriction. This fixes the normalization of the Casimir operators. Let $\rho := \frac{n-1}{2}$ and $c(\sigma) := \rho^2 - \sigma(\Omega_{\mathbf{M}})$. We define the operator $(A^{(s)})^2 := \Omega - c(\sigma)$. It gives rise to an unbounded selfadjoint operator on $L^2(X, V^{(s)})$. Let $A^{(s)}$ be its square root with non-negative real and imaginary part. In the super case V^s is a Dirac bundle carrying the Dirac operator D such that $A^s = |D|$.

We define the distributions $K(t) := \text{tr} \cos(tA)(x, x)$ and $J(t) := \text{tr} D \cos(tA^s)(x, x)$, where $x \in X$ is arbitrary. Applying Hadamard's analysis as in [4] we obtain the structure of these distributions at $t = 0$.

We first consider $K(t)$. Exactly as in [4], Corollary 3.2 we derive applying [11], Thm. 17.5.5.

Lemma 2.2 *The distribution $t \rightarrow K(t)$ has an asymptotic expansion at $t \rightarrow 0$ of the form*

$$K(t) = \sum_{k=-(n-1)/2}^0 c_{2k} \delta^{(2k)}(t) + \sum_{k=0}^{\infty} c_{2k+1} |t|^{2k+1}.$$

In fact, we will show that $c_k = 0$ for $k > 0$ employing the analysis on the dual symmetric space $S^n = X_d = \mathbf{G}^d/\mathbf{K}$, i.e. the unit sphere. Let $V_d \rightarrow X_d$ be the homogeneous bundle associated to the lift γ carrying the operator A_d . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition of \mathfrak{g} , α be the root of $(\mathfrak{a}, \mathfrak{n})$ and $H_\alpha \in \mathfrak{a}$ be the root vector satisfying $\alpha(H_\alpha) = 1$. Define $\epsilon(\sigma) \in \{0, 1/2\}$ by the condition : $e^{2\pi i \epsilon(\sigma)} = \sigma(2\pi i H_\alpha) \in \{\pm 1\}$. We consider the lattice $L := \epsilon(\sigma) + \mathbf{Z}$ and the even Weyl polynomial $P(\lambda) := P(\lambda, \sigma)$ ([4]). In order to define $P(\lambda, \sigma)$ let $\mathfrak{m} \subset \mathfrak{k}$ be the centralizer of \mathfrak{a} and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \subset \mathfrak{m} \oplus \mathfrak{a}$ be a Cartan subalgebra of \mathfrak{g} . Then \mathfrak{t} is a Cartan subalgebra of \mathfrak{m} . Fix a positive root system $\Phi^+(\mathfrak{g}, \mathfrak{h})$ compatible with the choice of α , let $\delta := \frac{1}{2} \sum_{\beta \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \beta$ and $\rho_m := \delta - \frac{n-1}{2} \alpha$. Then

$$P(\lambda, \sigma) := \prod_{\beta \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \frac{(\lambda + \mu_\sigma + \rho_m, \beta)}{(\delta, \beta)},$$

where $\mu_\sigma \in \mathfrak{t}^*$ is the highest weight of σ . Note that $\epsilon(w\sigma) = \epsilon(\sigma)$ and $P(\lambda, w\sigma) = P(\lambda, \sigma)$.

We consider weighted dimensions of eigenspaces since γ is a virtual representation. Let $m_d(\lambda, \gamma, \sigma) := \text{Tr} E_{A_d}(\{\lambda\})$ be the multiplicity of the eigenvalue $\lambda \in \mathbf{R}$. Moreover, for $0 \leq \lambda \in L$ let $m_d(\lambda) := m_d(\lambda, \sigma) := P(\lambda, \sigma)$ in the case (a) and $m_d(\lambda) := m_d(\lambda, \sigma) := 2P(\lambda, \sigma)$ in the case (b). From [4], Proposition 9.3 we obtain

Lemma 2.3 *Essentially, the eigenvalues of A_d form the ladder $\{0 \leq \lambda \in L\}$ and have the multiplicities $m_d(\lambda, \gamma, \sigma) = m_d(\lambda)$. There are at most finitely many exceptional eigenvalues or gaps of finite multiplicity.*

Now we compute the distribution $K_d(t) := \text{Tr} \cos(tA_d)$. Note the equation

$$\sum_{k=0}^{\infty} \cos(tk) = \pi\delta(t) + 1/2.$$

Thus, if $\epsilon(\sigma) = 0$, then

$$\tilde{K}_d(t) := d \sum_{k=0}^{\infty} P(k) \cos(tk) = d\pi P(\imath \frac{d}{dt}) \delta(t) + dP(0)/2$$

(note that $P(\lambda)$ is even), where $d = 1$ in the case (a) and $d = 2$ in the case (b). Also

$$\begin{aligned} \sum_{k=0}^{\infty} \cos((k+1/2)t) &= \cos(t/2) \sum_{k=0}^{\infty} \cos(kt) - \sin(t/2) \sum_{k=0}^{\infty} \sin(kt) \\ &= \pi\delta(t) + \frac{1}{2}\cos(t/2) - \frac{1}{2}\cos(t/2) \\ &= \pi\delta(t) \end{aligned}$$

Hence in the case $\epsilon(\sigma) = 1/2$

$$\tilde{K}_d(t) := d \sum_{k=0}^{\infty} P(k+1/2) \cos((k+1/2)t) = d\pi P(\imath \frac{d}{dt}) \delta(t).$$

Applying Hadamard's analysis [11], Thm. 17.5.5 to the wave operator $\cos(tA_d)$ we get an asymptotic expansion

$$K_d(t) = \omega_{n+1} \sum_{k=-(n-1)/2}^0 c_{d,2k} \delta^{(2k)}(t) + \sum_{k=0}^{\infty} c_{d,2k+1} |t|^{2k+1},$$

where $\omega_{n+1} := \text{vol}(S^n)$. Observe that by Proposition 2.3

$$K_d(t) - \tilde{K}_d(t) = \sum_{r=0}^{\text{finite}} c_r \cos(t\lambda_r),$$

where $c_r, \lambda_r \in \mathbf{C}$. Comparing this result with Lemma 2.2 we obtain:

Proposition 2.4 *We have*

$$K_d(t) = \pi P(\imath \frac{d}{dt}) \delta(t)$$

and $m_d(\lambda, \sigma, \gamma) = m_d(\lambda)$ for all $0 \neq \lambda \in L$ and in the case $\epsilon(\sigma) = 0$ we have $m_d(0, \sigma, \gamma) = dP(0)/2$.

Thus the unique lift $\gamma \in R(\mathbf{K})$ of $\sigma \in \hat{\mathbf{M}}$ is admissible in the sense of [4], Definition 2.2.

By [4], Lemma 3.6, we have $c_{d,k} = \imath c_k$. Thus $c_k = 0$ for $k > 0$ and we obtain

Corollary 2.5 *The distribution $K(t)$ is given by*

$$K(t) = d\pi\omega_{n+1}^{-1}P\left(\frac{d}{dt}\right)\delta(t).$$

Lemma 2.6 *The distribution $J(t)$ vanishes identically.*

Proof: In analogy to the derivation of 2.5 we must show the vanishing of the corresponding distribution $J_d(t)$ on the compact dual space. But the non-zero spectrum of D_d is symmetric, i.e. $\dim E_{D_d}(\{\lambda\}) = \dim E_{D_d}(\{-\lambda\})$, $\forall \lambda \neq 0$. \square

2.3 The hyperbolic contribution

The hyperbolic contribution is defined for semisimple $g \in \mathbf{G}$ to be the distribution

$$\begin{aligned} C_c^\infty(\mathbf{R}) \ni \phi &\rightarrow I_\phi(g) := \int_{\Omega_{<g>}} \text{tr} K_\phi(x, gx) dx \\ C_c^\infty(\mathbf{R}) \ni \phi &\rightarrow I_\phi^s(g) := \int_{\Omega_{<g>}} \text{tr} K_\phi^s(x, gx) dx, \end{aligned}$$

where $\Omega_{<g>}$ is the fundamental domain in X of the cyclic group generated by g and $K_\phi^{(s)}(x, y)$ is the kernel of the smoothing, finite propagation operator

$$\begin{aligned} K_\phi &:= \int_{\mathbf{R}} \phi(t) \cos(tA) dt \\ K_\phi^s &:= \int_{\mathbf{R}} \phi(t) D \cos(tA^s) dt. \end{aligned}$$

We identify the fibres $V_x^{(s)}, V_{gx}^{(s)}$ by g using the homogeneous bundle structure of $V^{(s)}$.

Let $\mathbf{G} = \mathbf{KAN}$ be an adapted Iwasawa decomposition of \mathbf{G} such that $g = ma \in \mathbf{MA}^+$. Let $l(g) = |\log(a)|$ and

$$C(g, \sigma) := -\frac{l(g)e^{\rho l(g)} \text{tr} \sigma(m)}{2 \det(1 - \text{Ad}(ma)_{\mathbf{n}})}.$$

The proof of the following Proposition is similar to the one of Theorem 3.11 in [4].

Proposition 2.7 *The hyperbolic contribution $I_\phi(g)$ is given by*

$$\begin{aligned} I_\phi(g) &= C(g, \sigma)(\phi(l(g)) + \phi(-l(g))) \quad \text{case (a)} \\ I_\phi(g) &= (C(g, \sigma) + C(g, w\sigma))(\phi(l(g)) + \phi(-l(g))) \quad \text{case (b)}. \end{aligned}$$

Theorem 2.8 *The super hyperbolic contribution $I_\phi^s(g)$ is given by*

$$I_\phi^s(g) = \iota(C(g, \sigma) - C(g, w\sigma))(\phi'(l(g)) - \phi'(-l(g))),$$

where $\phi'(t)$ denotes the derivative of ϕ with respect to t .

Proof: We first identify the domain $\Omega_{<g>}$. We write $\mathbf{G} = \mathbf{AN}\mathbf{K}$ and $X = \mathbf{AN}$. Then it is easy to see that one can choose $\Omega_{<g>} := [1, a] \times \mathbf{N} \subset \mathbf{AN}$. Moreover, we trivialize the bundle

$$V^s = \mathbf{AN} \times V_{\gamma^s} \quad (5)$$

identifying $V_{\gamma^s} = V_o^s$, $o = [\mathbf{K}] \in X$. The kernel $K_\phi^s(x, y)$ becomes an $\text{End}(V_{\gamma^s})$ -valued function.

We first carry out the integration with respect to \mathbf{A} .

$$\begin{aligned} I_\phi^s(g) &= \int_0^a \int_{\mathbf{N}} \text{tr} \left(K_\phi^s(bn, ab\alpha_m(n)) \gamma^s(m) \right) db dn \\ &= |\log(a)| \int_{\mathbf{N}} \text{tr} \left(K_\phi^s(n, a\alpha_m(n)) \gamma^s(m) \right) dn \\ &= l(g) \int_{\mathbf{N}} \text{tr} \left(K_\phi^s(\alpha_{am}(n^{-1})n, a) \gamma^s(m) \right) dn. \end{aligned}$$

The map $h(ma) : n \rightarrow \alpha_{am}(n^{-1})n$ is a diffeomorphism of \mathbf{N} and the Haar measure dn transforms as (compare Helgason [10] 1.5.4)

$$dh(ma)(n) = -\det(1 - \text{Ad}(ma)_{\mathbf{n}})dn.$$

We continue the evaluation of $I_\phi^s(g)$ obtaining

$$I_\phi^s(g) = -\frac{l(g)}{\det(1 - \text{Ad}(ma)_{\mathbf{n}})} \int_{\mathbf{N}} \text{tr} \left(K_\phi^s(a^{-1}n, o) \gamma^s(m) \right) dn. \quad (6)$$

In order to evaluate the \mathbf{N} -integral we employ the Fourier transform of Helgason-type

$$F : C_c^\infty(X, V^s) \rightarrow C^\infty(\mathfrak{ia}^* \times \mathbf{K}, V_{\gamma^s}),$$

which is defined by

$$\langle v, F(f)(\lambda, k) \rangle := \int_X \langle f(x), \phi_{-\lambda, v}(k^{-1}x) \rangle_x dx.$$

Here $v \in V_{\gamma^s}^*$ and $\phi_{\lambda, v} \in C^\infty(X, V^s)$ is defined by $\phi_{\lambda, v}(an) := a^{\lambda+\rho}v$ with respect to the trivialization (5). The Fourier transform obviously extends to distributions with compact support $C_c^{-\infty}(X, V^s)$

$$F(f)(\lambda, k) := \langle f, l_k \phi_{-\lambda, v} \rangle,$$

where l_k is the left action by $k \in \mathbf{K}$. Let $-\Omega$ and $\Omega_{\mathbf{M}}$ be the Casimir operators of \mathbf{G} and \mathbf{M} . For $\lambda \in \mathfrak{ia}^*$ we have

$$\Omega \phi_{\lambda, v} = \phi_{\lambda, (|\lambda|^2 + |\rho|^2 - \Omega_{\mathbf{M}})v}.$$

This can be seen by using the decomposition $\mathbf{g} = \mathbf{m} \oplus \mathbf{a} \oplus (\mathbf{n} + \bar{\mathbf{n}})$. For $f \in C_c^\infty(X, V^s)$ we have

$$\begin{aligned} F((A^s)^2 f) &= (|\lambda|^2 + \rho^2 - \Omega_{\mathbf{M}} - c(\sigma))F(f) \\ F(Df) &= \left(\sum_{i=1}^{n-1} c(X_i) \gamma'(Y_i) + \lambda c(H) \right) F(f). \end{aligned}$$

We explain the notation of the second line. We choose an orthonormal basis $(H, X_i, i = 1, \dots, n-1)$ of \mathfrak{p} with $H \in \mathbf{A}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition, $Y_i \in \mathfrak{k}$, $X_i + Y_i \in \mathfrak{n}$. Then $c(X_i)$ is the Clifford multiplication by X_i acting on the s -factor of γ^s .

As in the proof of [4], Thm. 3.11 we show that

$$F(D \cos(tA^s)f) = \left(\sum_{i=1}^{n-1} c(X_i) \gamma'(Y_i) + \lambda c(H) \right) \cos(t \sqrt{|\lambda|^2 + \rho^2 - \Omega_{\mathbf{M}} - c(\sigma)}) F(f). \quad (7)$$

Let I label the \mathbf{M} -types σ' occurring in V_{γ^s} . We decompose

$$V_{\gamma^s} = \oplus_{\sigma' \in I} V_{\gamma}(\sigma')$$

and let $P_{\sigma'}$ be the corresponding projections.

By the finite propagation speed of $D \cos(tA^s)$ and since ϕ has compact support we have

$$K_{\phi}^s : C_c^{\infty}(X, V^s) \rightarrow C_c^{\infty}(X, V^s).$$

Integrating (7) against $\phi(t)$ we obtain

$$F(K_{\phi}^s f) = \left(\sum_{i=1}^{n-1} c(X_i) \gamma'(Y_i) + \lambda c(H) \right) \sum_{\sigma' \in I} \frac{1}{2} (\hat{\phi}(d(\lambda, \sigma')) + \hat{\phi}(-d(\lambda, \sigma'))) P_{\sigma'} F(f),$$

where

$$d(\lambda, \sigma') := \sqrt{|\lambda|^2 + \rho^2 - \Omega_{\mathbf{M}} - c(\sigma')}$$

and

$$\hat{\phi}(s) = \int_{\mathbf{R}} \phi(t) e^{its} dt.$$

This equation extends to distributions $f \in C_c^{-\infty}(X, V^s)$. Inserting for f the delta distribution located in o with values in $V_{\gamma^s}^*$, we obtain

$$\begin{aligned} & F(K_{\phi}^s(., o))(\lambda, 1) \\ &= \left(\sum_{i=1}^{n-1} c(X_i) \gamma'(Y_i) + \lambda c(H) \right) \sum_{\sigma' \in I} \frac{1}{2} (\hat{\phi}(d(\lambda, \sigma')) + \hat{\phi}(-d(\lambda, \sigma'))) P_{\sigma'} \in \text{End}(V_{\gamma}^s). \end{aligned}$$

Now we can continue the evaluation of the hyperbolic contribution. Note that $K_{\phi}^s(a^{-1}n, o) \gamma^s(m)$ is a smooth $\text{End}(V_{\gamma^s})$ -valued function with compact support. We compute

$$\begin{aligned} & a^{-\rho} \int_N \text{tr} K_{\phi}^s(a^{-1}n, o) \gamma^s(m) \\ &= \frac{1}{2\pi i} \int_{\mathfrak{ia}^*} \int_{\mathbf{A}} e^{\lambda(\log(a) + \log(b))} b^{\rho} \int_N \text{tr} K_{\phi}^s(bn, o) \gamma^s(m) dn db d\lambda \\ &= \frac{1}{2\pi i} \int_{\mathfrak{ia}^*} e^{\lambda(\log(a))} \text{tr} F(K_{\phi}^s(., o))(-\lambda, 1) \gamma^s(m) d\lambda \\ &= \frac{1}{2\pi i} \int_{\mathfrak{ia}^*} e^{\lambda(\log(a))} \text{tr} \left(\sum_{i=1}^{n-1} c(X_i) \gamma'(Y_i) - \lambda c(H) \right) \\ & \quad \sum_{\sigma' \in I} \frac{1}{2} (\hat{\phi}(d(\lambda, \sigma')) + \hat{\phi}(-d(\lambda, \sigma'))) P_{\sigma'} \gamma^s(m) d\lambda \end{aligned}$$

Note that $r(\gamma^s) = (s^+ \otimes r(\gamma')) \oplus (s^- \otimes r(\gamma'))$ as an \mathbf{M} -module and $\sum_{i=1}^{n-1} c(X_i)\gamma'(Y_i)$ intertwines these two summands. Thus this sum does not contribute to the trace. $c(H)$ acts by $\pm i$ on s^\pm . Using $(s^+ - s^-) \otimes r(\gamma') = \sigma - w\sigma$ we obtain further

$$\begin{aligned} & a^{-\rho} \int_N \text{tr} K_\phi^s(a^{-1}n, o) \gamma^s \\ &= -\frac{1}{2\pi} \int_{\mathfrak{ia}^*} \lambda e^{\lambda(\log(a))} \frac{1}{2} (\hat{\phi}(-|\lambda|) + \hat{\phi}(|\lambda|)) \text{tr}(\sigma(m) - w\sigma(m)) d\lambda \\ &= \frac{i}{2} (\phi'(l(g)) - \phi'(-l(g))) \text{tr}(\sigma(m) - w\sigma(m)), \end{aligned}$$

where $\phi'(t) = \frac{d}{dt}\phi(t)$. The contribution of the \mathbf{M} -types different from σ and $w\sigma$ cancels out and $\sigma, w\sigma$ contribute with multiplicity 1, -1 . We obtain the expression for $I_\phi^s(g)$ claimed in the Theorem by inserting the last line into (6). \square

2.4 The distributional trace formula

Let $M = \Gamma \backslash \mathbf{G}/\mathbf{K}$ be an odd-dimensional, closed, hyperbolic manifold. Let $\chi : \Gamma \rightarrow \text{End}(E)$ be a finite-dimensional unitary representation of Γ and E_M be the corresponding flat bundle over M . Let $r := \dim(E)$. Let $(A_M^{(s)})^2, D_M$ be the operators associated to $\sigma \in \hat{\mathbf{M}}$ and twisted with E acting on sections of $V_M \otimes E_M \rightarrow M$. Define $A_M^{(s)}$ as the square root of the selfadjoint extension of $(A_M^{(s)})^2$ with non-negative real and imaginary part. We will suppress the χ in our notation, where it is possible. Let $\phi \in C_c^\infty(\mathbf{R})$. Then

$$\begin{aligned} K_{M,\phi} &:= \int_{-\infty}^{\infty} \phi(t) \cos(tA_M) dt \\ K_{M,\phi}^s &:= \int_{-\infty}^{\infty} \phi(t) D_M \cos(tA_M^s) dt \end{aligned}$$

are of trace class. The linear maps

$$C_c^\infty(\mathbf{R}) \ni \phi \rightarrow \text{Tr} K_{M,\phi}^{(s)}$$

define distributions on \mathbf{R} formally written as

$$\text{Tr} \cos(tA_M), \quad \text{Tr} D_M \cos(tA_M^s).$$

Let $n_\Gamma(g)$ be the number of classes in $\Gamma_g / \langle g \rangle$, where Γ_g is the centralizer of g in Γ and $\langle g \rangle$ is the group generated by g . By CT we denote the set of conjugacy classes of Γ .

Proposition 2.9 *We have the following equations of distributions.*

$$\begin{aligned} & \text{Tr} \cos(tA_M) \\ &= \text{vol}(M) \pi r \omega_{n-1}^{-1} P\left(\frac{d}{dt}\right) \delta(t) \end{aligned}$$

$$\begin{aligned}
& + \sum_{[g] \in C\Gamma, [g] \neq [1]} \frac{C(g, \sigma) \text{tr } \chi(g)}{n_\Gamma(g)} (\delta(t - l(g)) + \delta(-t - l(g))) \quad \text{case (a),} \\
& \text{Trcos}(tA_M) \\
& = \text{vol}(M) 2\pi r \omega_{n+1}^{-1} P\left(\frac{d}{dt}\right) \delta(t) \\
& + \sum_{[g] \in C\Gamma, [g] \neq [1]} \frac{(C(g, \sigma) + C(g, w\sigma)) \text{tr } \chi(g)}{n_\Gamma(g)} (\delta(t - l(g)) + \delta(-t - l(g))) \quad \text{case (b),} \\
& \text{TrDcos}(tA_M^s) \\
& = \sum_{[g] \in C\Gamma, [g] \neq [1]} -i \frac{(C(g, \sigma) - C(g, w\sigma)) \text{tr } \chi(g)}{n_\Gamma(g)} (\delta'(t - l(g)) - \delta'(t + l(g)))
\end{aligned}$$

The proof is analogous to the one of [4], Prop 3.12. The twist χ is incorporated by noting that

$$K_{M, \phi}(x, x) = \sum_{g \in \Gamma} K_\phi(\tilde{x}, g\tilde{x}) \otimes \chi(g),$$

where $\tilde{x} \in X$ is a preimage of $x \in M$. In the super case the contribution of the identity vanishes.

3 Theta functions

Let $M = \Gamma \backslash \mathbf{G}/\mathbf{K}$ be a closed, odd-dimensional hyperbolic manifold, $\sigma \in \hat{\mathbf{M}}$ and χ be a finite dimensional unitary representation of Γ . Let $\gamma \in R(\mathbf{K})$ be the lift of σ and $(A_M^{(s)})^2$ be the associated operator on M twisted with χ . Define $A_M^{(s)}$ as the square root of $(A_M^{(s)})^2$ with non-negative real and imaginary part. In the case (b) let D_M be the corresponding Dirac operator twisted with χ .

Definition 3.1 For $\text{Re}(t) > 0$ we define the theta function associated to M , σ and χ by

$$\theta(t) := \theta_\chi(t, \sigma) := \text{Tre}^{-tA_M}.$$

In the super case (b) we also define the super theta function

$$\theta^s(t) := \theta_\chi^s(t, \sigma) := \text{Trsign}(D_M) e^{-tA_M^s}.$$

The theta functions are holomorphic functions on the right half plan $\text{Re}(t) > 0$. The goal of the present section is to provide their meromorphic extensions and to discuss their singularity picture. We will suppress the σ and χ in our notation.

Theorem 3.2 *The theta function $\theta(t)$ admits a meromorphic continuation to the whole complex plane. It satisfies the functional equation*

$$\theta(t) + \theta(-t) = 0.$$

The singularities of $\theta(t)$ are first order poles at $t = \pm il(g)$, $g \in C\Gamma$ with residue

$$res_{t=\pm il(g)}\theta(t) = \begin{cases} \frac{C(g,\sigma)tr \chi(g)}{\pi n_\Gamma(g)} & \text{case (a)} \\ \frac{(C(g,\sigma)+C(g,w\sigma))tr \chi(g)}{\pi n_\Gamma(g)} & \text{case (b)} \end{cases}$$

and a pole of order $n-1$ at $t=0$. The super theta function has a meromorphic continuation to the whole complex plane. It is regular at $t=0$ and satisfies

$$\theta^s(t) - \theta^s(-t) = 0, \quad \theta(0) = \eta(D),$$

where $\eta(D)$ is the eta invariant of the Dirac operator D . The singularities of $\theta^s(t)$ are first order poles at $t = \pm il(g)$, $g \in C\Gamma$ with residue

$$res_{t=\pm il(g)}\theta(t) = \pm \frac{(C(g,\sigma) - C(g,w\sigma))tr \chi(g)}{\pi n_\Gamma(g)}.$$

The properties of the super theta function were already obtained by Juhl [12], Satz 9.1.4.

Proof: The proof for the non-super theta functions goes exactly as in [4], Thm. 4.6. The only modification is that we define θ on the half plane $Re(t) < 0$ using the new functional equation, i.e. $\theta(-t) := -\theta(t)$, $Re(t) > 0$. Note that the identity contribution of the trace formula $K(t)$ is localized at $t=0$. The singularity of θ at $t=0$ is again given by the local pseudodifferential analysis as in [7].

We now discuss the super case. We first derive a meromorphic continuation of the derivative $(\theta^s)'(t)$. It turns out that this derivative has second order poles with vanishing residues and can thus be integrated to give θ^s .

Since $D_M = sign(D_M)A_M^s$ we have $(\theta^s)'(t) = -Tr D_M e^{-tA_M^s}$. We try to continue $(\theta^s)'(t)$ using the functional equation

$$(\theta^s)'(t) + (\theta^s)'(-t) = 0. \tag{8}$$

We define $(\theta^s)'$ as a distribution on \mathbf{C} . Let $\phi \in C_c^\infty(\mathbf{C})$. Then by definition

$$\langle (\theta^s)', \phi \rangle := \lim_{\epsilon \rightarrow 0} \int_{|Re(v)| \geq \epsilon} (\theta^s)'(v) \phi(v) dv.$$

We compute the distributional derivative $\bar{\partial}(\theta^s)'$. Note that $(\theta^s)'(it + \epsilon)$ converges (considered as a distribution with respect to t) to a tempered distribution on \mathbf{R} when $\epsilon \rightarrow 0$.

$$\begin{aligned} & \langle \bar{\partial}(\theta^s)', \phi \rangle \\ &= \langle (\theta^s)', \bar{\partial}^* \phi \rangle \\ &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \left[-\frac{\partial}{\partial u} - \frac{\partial}{\partial(it)} \right] (\phi(it + u) + \phi(-it - u)) (\theta^s)'(it + u) du \, dt \\ &= \langle \frac{1}{2} ((\theta^s)'(it + 0) + (\theta^s)'(-it + 0)), \phi(it) \rangle. \end{aligned}$$

The distributional trace formula states

$$\begin{aligned}
& (\theta^s)'(it + 0) + (\theta^s)'(-it + 0) \\
&= -2\text{Tr } D_M \cos(tA_M^s) \\
&= 2i \sum_{[g] \in C\Gamma, [g] \neq [1]} \frac{(C(g, \sigma) - C(g, w\sigma)) \text{tr } \chi(g)}{n_\Gamma(g)} (\delta'(t - l(g)) - \delta'(t + l(g))).
\end{aligned}$$

Thus,

$$\langle \bar{\partial}(\theta^s)', \phi \rangle = -i \sum_{[g] \in C\Gamma, [g] \neq [1]} \frac{(C(g, \sigma) - C(g, w\sigma)) \text{tr } \chi(g)}{n_\Gamma(g)} \left(\frac{d}{dt} \Big|_{t=l(g)} \phi(it) - \frac{d}{dt} \Big|_{t=-l(g)} \phi(it) \right).$$

Write $z = u + it$. Since

$$\bar{\partial} \frac{-i}{z^2} = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial(it)} \right) \frac{1}{z} = \pi \frac{\partial}{\partial t} \delta(z)$$

we obtain that $(\theta^s)'(t)$ is holomorphic near the imaginary axis except at $t = \pm il(g)$, $g \in C\Gamma$, where it has second order poles with the singular part

$$\pm \frac{(C(g, \sigma) - C(g, w\sigma)) \text{tr } \chi(g)}{\pi n_\Gamma(g)} \frac{1}{(t \mp il(g))^2}.$$

Thus we have provided a meromorphic continuation of $(\theta^s)'$ such that it only has second order poles with vanishing residues. Note that $(\theta^s)'(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$. Since the same holds for θ^s itself, we have

$$\theta^s(t) := \int_{-\infty}^t (\theta^s)'(u) du.$$

Thus θ^s is meromorphic and has first order poles with the residues claimed in the statement of the theorem. Integrating (8) we obtain

$$\theta^s(t) - \theta^s(-t) = 0.$$

Moreover,

$$\theta^s(0) = - \int_0^\infty (\theta^s)'(u) du \tag{9}$$

We now have to identify the expression (9) with $\eta(D)$. The η -invariant of D is defined as the value at $s = 0$ of the η -function

$$\eta(s) = \text{Tr} \frac{\text{sign}(D)}{|D|^s}.$$

Initially, the η -function is defined for $\text{Re}(s)$ large. But it has a meromorphic continuation to all of \mathbf{C} and turns out to be regular at $s = 0$ (Atiyah/Patodi/Singer [1]). We have

$$\eta(s) = \text{Tr} \frac{\text{sign}(D)}{|D|^s} = \frac{1}{\Gamma(s+1)} \int_0^\infty u^s \text{Tr } D e^{-u|D|} du.$$

Since $-Tr D e^{-u|D|} = (\theta^s)'(u)$ is regular at $u = 0$ we can let $s = 0$ obtaining

$$-\int_0^\infty (\theta^s)'(u) du = \eta(D).$$

□

4 Zeta functions

4.1 The logarithmic derivative

Let $M = \Gamma \backslash \mathbf{G}/\mathbf{K}$ be a closed, odd-dimensional hyperbolic manifold, $\sigma \in \hat{\mathbf{M}}$ and χ be a finite dimensional unitary representation of Γ . We say that $[g] \in C\Gamma$ is primitive, if $n_\Gamma(g) = 1$.

Definition 4.1 For $Re(s) > \rho = (n-1)/2$ we define the Selberg zeta function associated to σ by the Euler product.

$$Z_{S,\chi}(s, \sigma) = \prod_{[g] \in C\Gamma, [g] \neq 1, \text{primitive}} \prod_{k=0}^{\infty} \det \left(1 - e^{(-\rho-s)l(g)} S^k(Ad(g)_{\mathbf{n}}^{-1}) \otimes \sigma(m) \otimes \chi(g) \right).$$

We will again suppress σ and χ in our notation, where it is possible. In the case (b) we study $Z_S(s, \sigma)$ via $S(s) := S(s, \sigma) := Z_S(s, \sigma)Z_S(s, w\sigma)$ and $S^s(s) := S^s(s, \sigma) := Z_S(s, \sigma)/Z_S(s, w\sigma)$. $S^{(s)}$ are closely related to the analysis of the operator D_M .

In the case (a) the logarithmic derivative of $Z_S(s)$ is given by (see e.g. [4], Prop. 6.3)

$$D(p) := \sum_{[g] \in C\Gamma, [g] \neq 1} \frac{2C(g, \sigma) tr \chi(g) e^{-pl(g)}}{n_\Gamma(g)}.$$

In the case (b) the logarithmic derivatives of $S^{(s)}(s)$ are given by

$$\begin{aligned} D(p) &:= \sum_{[g] \in C\Gamma, [g] \neq 1} 2 \frac{(C(g, \sigma) + C(g, w\sigma)) tr \chi(g) e^{-pl(g)}}{n_\Gamma(g)} \\ D^s(p) &:= \sum_{[g] \in C\Gamma, [g] \neq 1} 2 \frac{(C(g, \sigma) - C(g, w\sigma)) tr \chi(g) e^{-pl(g)}}{n_\Gamma(g)}. \end{aligned}$$

We use the Ψ -construction as in [4]. Let $f(p)$ be a function of one complex variable and $N \in \mathbf{N}$. Then $\Psi(f(p))$ is a function of N complex variables p_1, \dots, p_N . It is defined if $p_i \neq p_j, \forall i \neq j$. For example, consider the function e^{pt} . Then

$$\Psi(e^{pt}) := \sum_{i=1}^N \left(\prod_{j=1, j \neq i}^N \frac{1}{p_j^2 - p_i^2} e^{p_i t} \right).$$

Let $R(p^2) := (A_M^2 + p^2)^{-1}$ and $L < \infty$ with $-L < A_M^2$. Using our distributional trace formulas we can (similar to [4], Prop.5.5) prove

Proposition 4.2 (*D* and the trace of resolvents) For $N \geq n/2 + 1$ and $p_1, \dots, p_N \in \mathbf{C}$ with $\text{Re}(p_i) > L$ and $p_i \neq p_j$ for all $i \neq j$ we have in the non-super case

$$\text{Tr} \prod_{i=1}^N R(p_i^2) = d\pi \text{vol}(M) \omega_{n+1}^{-1} r \Psi\left(\frac{P(p)}{p}\right) + \Psi\left(\frac{D(p)}{2p}\right).$$

In the super case of (b) we have

$$\text{Tr } D_M \prod_{i=1}^N R(p_i^2) = -i \Psi\left(\frac{D^s(p)}{2}\right).$$

Proof: We discuss first the non-super case. Consider the function

$$f(pt) = \frac{1}{2} \begin{cases} e^{-pt} & t > 0 \\ -e^{pt} & t \leq 0 \end{cases}.$$

If $\text{Re}(p_i) > 0, \forall i = 1, \dots, N$, then $\Psi(f(pt))$ vanishes exponentially if $t \rightarrow \pm\infty$. Moreover, it is anti-symmetric with respect to t and smooth for $t \neq 0$. At $t = 0$ it has continuous derivatives up to the order $2N - 1$, since the Ψ kills the first $N - 1$ even Taylor coefficients [4], Lemma 5.4.

Note that

$$\begin{aligned} \prod_{i=1}^N R(p_i^2) &= \int_0^\infty \Psi(e^{-pt}) \frac{\sin(tA_M)}{A_M} dt \\ &= \int_{-\infty}^\infty \Psi(f(pt)) \frac{\sin(tA_M)}{A_M} dt. \end{aligned}$$

Integrating the distributional trace formula for $\text{Tr} \cos(tA_M)$ we obtain

$$\begin{aligned} &\text{Tr} \frac{\sin(tA_M)}{A_M} \\ &= \text{vol}(M) \omega_{n+1}^{-1} \pi r P\left(\frac{d}{dt}\right) (\Theta(t) - 1/2) \\ &\quad + \sum_{[g] \in \text{CT}, [g] \neq [1]} \frac{C(g, \sigma) \text{tr } \chi(g)}{n_\Gamma(g)} (\Theta(t - l(g)) - \Theta(-t - l(g))) \quad \text{case (a),} \\ &\text{Tr} \frac{\sin(tA_M)}{A_M} \\ &= \text{vol}(M) \omega_{n+1}^{-1} 2\pi r P\left(\frac{d}{dt}\right) (\Theta(t) - 1/2) \\ &\quad + \sum_{[g] \in \text{CT}, [g] \neq [1]} \frac{(C(g, \sigma) + C(g, w\sigma)) \text{tr } \chi(g)}{n_\Gamma(g)} (\Theta(t - l(g)) - \Theta(-t - l(g))) \quad \text{case (b),} \end{aligned}$$

where the integration constant was fixed such that the right hand side is anti-symmetric. We can insert $\Psi(f(pt))$ into the trace formula. In order to justify this, note that one

can apply the trace formula to an anti-symmetric version $\Psi(\tilde{f}(pt))$ smoothed at $t = 0$ such that $\Psi(\tilde{f}(pt)) = \Psi(f(pt))$ for $|t| > 1$. Now let a sequence of such $\Psi(\tilde{f}(pt))$ tend to $\Psi(f(pt))$ in $C_{loc}^{2N-1}(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} \Psi(\tilde{f}(pt)) \frac{\sin(tA_M)}{A_M} dt \rightarrow \int_{-\infty}^{\infty} \Psi(f(pt)) \frac{\sin(tA_M)}{A_M} dt$$

in the sense of trace class operators. Moreover, all individual terms in the trace formula converge such that their sum converges, as well. We obtain

$$\begin{aligned} & Tr \prod_{i=1}^N R(p_i^2) \\ &= d\pi r \omega_{n+1}^{-1} vol(M) \int_{-\infty}^{\infty} (P(\frac{d}{dt})(\Theta(t) - 1/2)) \Psi(f(pt)) dt \\ &+ \sum_{[g] \in CT, [g] \neq 1} \frac{\left\{ \begin{array}{ll} C(g, \sigma) tr \chi(g) & (a) \\ (C(g, \sigma) + C(g, w\sigma)) tr \chi(g) & (b) \end{array} \right\}}{n_{\gamma}(g)} \Psi\left(\frac{f(pl(g))}{p}\right) \end{aligned}$$

The second term is nothing else then

$$\Psi\left(\frac{D(p)}{2p}\right)$$

while the first term gives

$$\begin{aligned} & 2d\pi r \omega_{n+1}^{-1} vol(M) \int_0^{\infty} P(\frac{d}{dt}) \Psi(f(pt)) dt \\ &= d\pi r \omega_{n+1}^{-1} vol(M) \int_0^{\infty} P(\frac{d}{dt}) \Psi(e^{-pt}) dt \\ &= d\pi r \omega_{n+1}^{-1} vol(M) \int_0^{\infty} \Psi(P(p)e^{-pt}) dt \\ &= d\pi r \omega_{n+1}^{-1} vol(M) \Psi\left(\frac{P(p)}{p}\right) \end{aligned}$$

In the super case we argue similarly. We have

$$D_M \prod_{i=1}^N R(p_i^2) = \int_{-\infty}^{\infty} \Psi(f(pt)) D_M \frac{\sin(tA_M)}{A_M} dt.$$

Applying the integrated distributional trace formula

$$Tr D_M \frac{\sin(tA_M)}{A_M} = -\imath \sum_{[g] \in CT, [g] \neq [1]} \frac{C(g, \sigma) - C(g, w\sigma)}{n_{\Gamma}(g)} (\delta(t - l(g)) - \delta(t + l(g)))$$

we obtain

$$\begin{aligned} \text{Tr } D_M \prod_{i=1}^N R(p_i^2) &= \sum_{[g] \in C\Gamma, [g] \neq 1} -\imath \frac{(C(g, \sigma) - C(g, w\sigma)) \text{tr } \chi(g)}{n_\Gamma(g)} \Psi(e^{-pl(g)}) \\ &= -\imath \Psi\left(\frac{D^s(p)}{2}\right). \end{aligned}$$

□

Corollary 4.3 *The logarithmic derivative of the Selberg zeta function $D^{(s)}(p)$ has an analytic continuation to all of \mathbf{C} . Its singularities are first order poles at $\pm \imath \lambda$, $\lambda \in \text{spec } A_M$ with residue $\dim E_{A_M}(\{\lambda\})$ ($2\dim E_{A_M}(\{0\})$ if $\lambda = 0$) in the non-super case and at $\imath \lambda$, $\lambda \in \text{spec } D_M$, with residue $\dim E_{D_M}(\{\lambda\}) - \dim E_{D_M}(\{-\lambda\})$ in the super case.*

4.2 The fundamental properties of the zeta functions

All residues of the logarithmic derivatives of $D(p)^{(s)}$ are integers. Hence we can apply $\exp \circ \int_s^\infty dp$ to $D^{(s)}(p)$ in order to obtain the continuation of the Selberg zeta functions.

Proposition 4.4 *$Z_S(s)$ in the case (a) and $S(s)$ in the case (b) have analytic continuations to all of \mathbf{C} . Their singularities (zeros and poles) are at $\pm \imath \lambda$, $\lambda \in \text{spec } A_M$ of order $\dim E_{A_M}(\{\lambda\})$ ($2\dim E_{A_M}(\{0\})$ if $\lambda = 0$). In the case (b) $S^s(s)$ has its singularities at $\imath \lambda$, $\lambda \in \text{spec } D_M$ of order $\dim E_{D_M}(\{\lambda\}) - \dim E_{D_M}(\{-\lambda\})$.*

We derive the functional equations of the zeta functions.

Proposition 4.5 *The functional equations of the Selberg zeta functions are*

$$\begin{aligned} \frac{Z_S(s)}{Z_S(-s)} &= \exp\left(2\pi r \omega_{n+1}^{-1} \text{vol}(M) \int_0^s P(p) dp\right) \text{ case (a)} \\ \frac{S(s)}{S(-s)} &= \exp\left(4\pi r \omega_{n+1}^{-1} \text{vol}(M) \int_0^s P(p) dp\right) \text{ case (b)}. \end{aligned}$$

The zeta function $S^s(s)$ satisfies the functional equation

$$S^s(s) S^s(-s) = e^{2\pi \imath \eta(D_M)}.$$

Moreover $S^s(0) = e^{\imath \pi \eta(D_M)}$.

The results on S^s were obtained in the special case of the signature operator by Millson [14]. They also follow by a specialization of results of Moscovici/Stanton [15] as explained in Juhl [12]. *Proof:* We first consider the non-super case. From Proposition 4.2 we obtain by inserting $p_1 = \pm p$ and summing

$$\begin{aligned} D(p) + D(-p) &= d\pi r \omega_{n+1}^{-1} \text{vol}(M) P(p) \\ D^s(p) - D^s(-p) &= 0. \end{aligned}$$

Integrating and exponentiating the first equation we obtain the desired result. In the case (b) the functional equation for $S^s(s)$ is more interesting since the integration constant does not drop out. Note that

$$R(p^2) = \int_0^\infty e^{-p^2 t} e^{-t(A_M^s)^2} dt.$$

Hence, assuming $p_i \gg 0, \forall i = 1, \dots, N$, we obtain

$$\begin{aligned} \Psi(D^s(p)) &= 2i \operatorname{Tr} D_M \prod_{i=1}^N R(p_i^2) \\ &= 2i \operatorname{Tr} D_M \Psi(R(p^2)) \\ &= 2i \operatorname{Tr} \int_0^\infty \Psi(e^{-p^2 t}) D_M e^{-t(A_M^s)^2} dt. \end{aligned}$$

Since $\operatorname{Tr} D_M e^{-t(A_M^s)^2}$ is regular near $t = 0$ (and vanishes there) and $D^s(p)$ vanishes at infinity, we can deduce

$$\begin{aligned} D^s(p) &= 2i \int_0^\infty e^{-p^2 t} \operatorname{Tr} D_M e^{-t(A_M^s)^2} dt \\ \log S^s(s) &= 2i \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{t}} (1 - \Phi(t^{1/2}s)) \operatorname{Tr} D_M e^{-t(A_M^s)^2} dt, \end{aligned}$$

where $\Phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$ is the error integral. Hence

$$\begin{aligned} \log S^s(s) + \log S^s(-s) &= 2i \int_0^\infty \frac{\sqrt{\pi}}{\sqrt{t}} \operatorname{Tr} D_M e^{-t(A_M^s)^2} dt \\ &= 2\pi i \eta(D_M). \end{aligned}$$

The functional equation follows. \square

Proposition 4.6 *In the case (b) the zeta function $Z_S(s) := Z_S(s, \sigma)$ has a meromorphic continuation to the whole complex plane. It has singularities at $0 \neq s = i\lambda$ of order*

$$\frac{1}{2} (\dim E_{A_M}(\{\lambda\}) + \dim E_{D_M}(\{\lambda\}) - \dim E_{D_M}(\{-\lambda\}))$$

and of order $\dim \ker A_M$ at $s = 0$. $Z_S(s)$ satisfies the functional equation

$$\frac{Z_S(s, \sigma)}{Z_S(-s, w\sigma)} = e^{\pi i \eta(D_M)} \exp \left(2\pi i \omega_{n+1}^{-1} \operatorname{vol}(M) \int_0^s P(p) dp \right).$$

Proof: For $\operatorname{Re}(s) > \rho$ we have $Z_S(s) = \sqrt{S(s)S^s(s)}$. The continuation of $Z_S(s)$ is obtained from that of $S^{(s)}(s)$ if we can take the square root. It is sufficient to show that all singularities of $S(s)S^s(s)$ have even order. We choose the representative γ of the lift of

$\sigma + w\sigma$ such that $V(\gamma) = V(\gamma^s)$ if one forgets the \mathbf{Z}_2 -grading. In particular, $\lambda \in \text{spec}|D_M|$ iff $\lambda \in \text{spec } A_M$. For $\lambda > 0$ we have $\dim E_{D_M}(\{\lambda\}) - \dim E_{D_M}(\{-\lambda\}) = \dim E_{A_M}(\{\lambda\}) \pmod{2}$. Hence the order of the singularity at $s \neq 0$ of $S(s)S^s(s)$ is even by Proposition 4.4. At $s = 0$ the function $S^s(s)$ is regular and $S(s)$ has a singularity of even order. Thus we can take the square root above and obtain a meromorphic continuation of $Z_S(s)$ to all of \mathbf{C} . The order singularity of $Z_S(s)$ at $0 \neq s = \imath\lambda$ is $\frac{1}{2}(\dim E_{A_M}(\{\lambda\}) + \dim E_{D_M}(\{\lambda\}) - \dim E_{D_M}(\{-\lambda\}))$ and $\dim \ker A_M$ at $s = 0$. We also obtain

$$\begin{aligned} \frac{Z_S(s, \sigma)}{Z_S(-s, w\sigma)} &= \sqrt{\frac{S(s, \sigma)S^s(s, \sigma)}{S(-s, w\sigma)S^s(-s, w\sigma)}} \\ &= \sqrt{\frac{S(s, \sigma)}{S(-s, \sigma)}S^s(s, \sigma)S^s(-s, \sigma)} \\ &= e^{\pi\eta(D_M)} \exp\left(2\pi r\omega_{n+1}^{-1} \text{vol}(M) \int_0^s P(p)dp\right) \end{aligned}$$

□

Now we relate the zeta functions to zeta-regularized determinants.

Proposition 4.7 *We have*

$$\begin{aligned} Z_S(s) &= \det(s^2 + A_M^2) e^{2\pi r\omega_{n+1}^{-1} \text{vol}(M) \int_0^s P(p)dp} \\ S(s) &= \det(s^2 + A_M^2) e^{4\pi r\omega_{n+1}^{-1} \text{vol}(M) \int_0^s P(p)dp}. \end{aligned}$$

Proof: For $|Re(p_i)|$ large we consider the function L_M of p_i , $i = 1, \dots, N$:

$$L_M(p_1, \dots, p_N) := \text{Tr } \Psi\left(\frac{1}{p^2 + A_M^2}\right).$$

We can write

$$\begin{aligned} L_M(p_1, \dots, p_N) &= \text{Tr} \Psi\left(\int_0^\infty e^{-t(p^2 + A_M^2)} dt\right) \\ &= \text{Tr} \Psi\left(\int_0^\infty -\frac{d}{2pd} e^{-t(p^2 + A_M^2)} \frac{dt}{t}\right) \\ &= \Psi\left(-\frac{d}{2pd} P.F. \int_0^\infty \text{Tr} e^{-t(p^2 + A_M^2)} \frac{dt}{t}\right), \end{aligned}$$

where $P.F. \int_0^\infty \dots$ stands for taking the finite part of $\int_\epsilon^\infty \dots$

Applying Soule/Abramovich/Burnol/Kramer, [19], Theorem 5.1.1, we obtain

$$L_M(p_1, \dots, p_N) = \Psi\left(\frac{d}{2pd} \ln(\det(p^2 + A_M^2))\right). \quad (10)$$

The determinant is the zeta regularized (super) determinant of $p^2 + A_M^2$ defined by

$$\ln(\det(p^2 + A_M^2)) := -P.F. \int_0^\infty \text{Tr} e^{-t(p^2 + A_M^2)} \frac{dt}{t}$$

(see the explanation on p.39 in [4]). Equation (10) extends analytically to all of \mathbf{C} . We obtain

$$\Psi\left(\frac{D(p) + 2d\pi r\omega_{n+1}^{-1}\text{vol}(M)P(p)}{2p}\right) = \Psi\left(\frac{d}{2p} \ln(\det(p^2 + A_M^2))\right).$$

It follows

$$D(p) + 2d\pi r\omega_{n+1}^{-1}\text{vol}(M)P(p) = R'(p) + \frac{d}{dp} \ln(\det(A_M^2 + p^2)),$$

where $R'(p)$ is a certain odd polynomial. Integrating once and exponentiating we obtain that the Selberg zeta function has the representation

$$Z(s) = e^{R(s)} \det(s^2 + A_M^2) e^{d\pi r\omega_{n+1}^{-1}\text{vol}(M) \int_{-s}^s P(p) dp},$$

where $Z(s)$ is $Z_S(s)$ ($d = 1$) or $S(s)$ ($d = 2$), respectively.

Lemma 4.8 $R(s) = 0$

Proof: Note that $\log D(s) \rightarrow 0$ exponentially as $s \rightarrow 0$. Let

$$\text{Tr} e^{-tA_M^2} \stackrel{t \rightarrow 0}{\sim} \sum_{k=0}^{\infty} c_{(2k-n)/2} t^{(2k-n)/2}$$

define the real numbers $c_{(2k-n)/2}$. Then by [20], Eq. 5.1,

$$-\log \det(s^2 + A_M^2) \stackrel{s \rightarrow 0}{\sim} \sum_{k=0}^{\infty} c_{(2k-n)/2} \Gamma((2k-n)/2) s^{n-2k}.$$

This asymptotic expansion containing only odd powers of s must cancel with the Plancherel term, i.e.

$$d\pi \text{vol}(M) \int_{-s}^s P(p) dp = \sum_{k=0}^{(n-1)/2} c_{(2k-n)/2} \Gamma((2k-n)/2) s^{n-2k},$$

$c_{(2k-n)/2} = 0$ for $2k > n$. It also follows $R(p) = 0$. \square

5 The Ruelle zeta function

5.1 Definition and relation with the Selberg zeta function

We discuss the Ruelle zeta function of compact odd dimensional hyperbolic manifolds M . Ruelle [18] introduced a zeta function for Anosov flows coding the length spectrum of closed orbits. In our situation the corresponding flow is the geodesic flow on the unit sphere bundle of M (twisted with a flat vector bundle). The closed orbits correspond to closed geodesics and thus to the conjugacy classes of Γ .

Definition 5.1 *The Ruelle zeta function of M associated to a r -dimensional unitary representation χ of Γ is defined by the infinite product*

$$Z_{R,\chi}(s) := \prod_{[g] \in CT, [g] \neq 1, \text{primitive}} \det(1 - \chi(g)e^{-sl(g)})^{-1}$$

converging for $\operatorname{Re}(s) > 2\rho$.

We consider the representation χ as fixed throughout this and the following subsection and shall omit the corresponding index.

Let σ^p be the p -th exterior powers of the complexified standard representation of $SO(n-1)$. For $p \neq (n-1)/2$ the representation σ^p is irreducible, while $\sigma^{\frac{n-1}{2}}$ splits into two non-equivalent irreducible subrepresentations $\sigma^{\frac{n-1}{2}} = \sigma^+ \oplus \sigma^-$, the spaces of selfdual and anti-selfdual forms. We have for the non trivial element w of the Weyl group $w\sigma^+ = \sigma^-$. Therefore it is natural to set

$$Z_S(s, \sigma^{\frac{n-1}{2}}) := S(s, \sigma^+) .$$

The following proposition is due to Fried ([8],[9]). One may also compare this with the computations for the even dimensional case in [4], 7.1, 8.1.

Proposition 5.2 *The Ruelle zeta function has the representation*

$$\begin{aligned} Z_R(s) &= \prod_{p=0}^{n-1} Z_S(s + \frac{n-1}{2} - p, \sigma^p)^{(-1)^p} \\ &= S(s, \sigma^+)^{(-1)^{\frac{n-1}{2}}} \prod_{p=0}^{\frac{n-3}{2}} \left(Z_S(s + \frac{n-1}{2} - p, \sigma^p) Z_S(s - (\frac{n-1}{2} - p), \sigma^p) \right)^{(-1)^p} . \end{aligned}$$

For the second equation one uses Poincaré duality.

The representation of the Ruelle zeta function provides its meromorphic continuation to the whole complex plane. That the Ruelle zeta function has a meromorphic continuation was observed in Ruelle's original work by dynamical methods ([18]).

5.2 The functional equation for the Ruelle zeta function

We employ the functional equations for the Selberg zeta functions in order to provide a functional equation for the Ruelle zeta function. We obtain

$$\begin{aligned} \frac{Z_R(s)}{Z_R(-s)} &= \left(\frac{S(s, \sigma^+)}{S(-s, \sigma^+)} \right)^{(-1)^{\frac{n-1}{2}}} \prod_{p=0}^{\frac{n-3}{2}} \left(\frac{Z_S(s + \frac{n-1}{2} - p, \sigma^p) Z_S(s - (\frac{n-1}{2} - p), \sigma^p)}{Z_S(-s + \frac{n-1}{2} - p, \sigma^p) Z_S(-s - (\frac{n-1}{2} - p), \sigma^p)} \right)^{(-1)^p} \\ &= \exp \left(\frac{2\pi r \operatorname{vol}(M)}{\omega_{n+1}} (2(-1)^{\frac{n-1}{2}} \int_0^s P(q, \sigma^+) dq + \right. \\ &\quad \left. + \sum_{p=0}^{\frac{n-3}{2}} (-1)^p \left(\int_0^{s + \frac{n-1}{2} - p} P(q, \sigma^p) dq + \int_0^{s - (\frac{n-1}{2} - p)} P(q, \sigma^p) dq \right) \right) . \end{aligned} \tag{11}$$

Let $h(s)$ denote the derivative of the exponent (up to the prefactor), i.e. the polynomial

$$h(s) = 2(-1)^{\frac{n-1}{2}} P(s, \sigma^+) + \sum_{p=0}^{\frac{n-3}{2}} (-1)^p (P(s + \frac{n-1}{2} - p, \sigma^p) + P(s - (\frac{n-1}{2} - p), \sigma^p)) .$$

Lemma 5.3 *The polynomial $h(s)$ is a constant*

$$h(s) \equiv n + 1 .$$

Proof: The claim is a consequence of identities in the Weyl polynomials. In order to establish them we employ a geometric argument. It is enough to evaluate $h(s)$ at a large number of points.

We consider the Grassmannian $B = SO(n+1)/SO(n-1) \times SO(2)$ of oriented 2-planes in \mathbf{R}^{n+1} as a homogeneous Kähler manifold. Each pair $(\sigma, k) \in SO(n-1)^\wedge \times \mathbf{Z} = (SO(n-1) \times SO(2))^\wedge$ defines a homogeneous holomorphic vector bundle $W(k, \sigma)$ on B .

The Borel-Weil-Bott theorem asserts that the representation of $SO(n+1)$ with the highest weight $\Lambda = \mu_\sigma + k\alpha \in \mathfrak{t}^* \oplus \mathfrak{a}^*$ ($k \gg 0$) can be realized as the space of holomorphic sections of $W(k, \sigma)$ and all higher cohomology groups of $W(k, \sigma)$ vanish. A consequence of the theorem of Borel-Weil-Bott is

$$P(k + \frac{n-1}{2}, \sigma) = \chi_a(B, W(k, \sigma)), \quad k \in \mathbf{Z},$$

where χ_a is the analytic genus of the bundle $W(k, \sigma)$, i.e. the Euler characteristic of the complex given by the Dolbeault resolution of $W(k, \sigma)$. Thus

$$P(k + \frac{n-1}{2}, \sigma) = \text{index}(\bar{\partial} + \bar{\partial}^*),$$

i.e. $P(k + \frac{n-1}{2}, \sigma)$ is the index of the Dirac type operator $\bar{\partial} + \bar{\partial}^*$ on the \mathbf{Z}_2 -graded vector bundle $\Lambda^{0,*} T^* B \otimes W(k, \sigma)$ (the grading given by even and odd form degree).

Note that $\Lambda^{p,0} T^* B \cong W(-p, \sigma^p)$. We obtain for $s \in \mathbf{Z}$ and $p \leq \frac{n-3}{2}$

$$P(s + \frac{n-1}{2} - p, \sigma^p) = \chi_a(\Lambda^{p,0} T^* B \otimes W(s, 1))$$

and

$$P(s - (\frac{n-1}{2} - p), \sigma^p) = \chi_a(\Lambda^{n-1-p,0} T^* B \otimes W(s, 1)) .$$

In addition,

$$2P(s, \sigma^+) = P(s, \sigma^+) + P(s, \sigma^-) = \chi_a(\Lambda^{\frac{n-1}{2},0} T^* B \otimes W(s, 1)) .$$

It follows that $h(s), s \in \mathbf{Z}$ is the index of $(\bar{\partial} + \bar{\partial}^*)$ on the bundle

$$\Lambda^{*,*} T^* B \otimes W(s, 1) = \Lambda^* T^* B \otimes W(s, 1)$$

graded by the total form degree. Now the differential operator $(\bar{\partial} + \bar{\partial}^*)$ can be deformed to $D = \nabla + \nabla^*$, where ∇ is induced by the Levi-Civita connection of B and the homogeneous connection on $W(s + \rho, 1)$. Here ∇ acts by alternating differentiation. The deformation is given by

$$D_t = (\bar{\partial} + \bar{\partial}^*) + t(\partial + \partial^*)$$

and stays inside the elliptic operators. ∂ is again defined using the connection. The index of D_t is independent of t and for $t = 1$ it is independent of the twisting line bundle and equal to $\text{index}(D_1) = \chi(B)$. The Euler characteristic of B can be computed using the orders of Weyl groups (see for example Bott [2])

$$\chi(B) = \frac{|W(SO(n+1))|}{|W(SO(n-1) \times SO(2))|} = \frac{|W(SO(n+1))|}{|W(SO(n-1))|} = \frac{2^{\frac{n-1}{2}}(\frac{n+1}{2})!}{2^{\frac{n-3}{2}}(\frac{n-1}{2})!} = n+1 \quad \square$$

Now we come back to discuss the functional equation of the Ruelle zeta function. $h(s) = n+1$ is the derivative of a polynomial $(n+1)s + C$ for some constant C . But the left hand side of (11) evaluated at $s = 0$ is 1. We conclude that $C = 0$ and obtain

Theorem 5.4 *The Ruelle zeta function satisfies the functional equation*

$$\frac{Z_R(s)}{Z_R(-s)} = e^{\frac{2\pi r(n+1)\text{vol}(M)}{\omega_{n+1}}s}.$$

5.3 Analytic torsion and the Ruelle zeta function

We derive the result of Fried [8] that was generalized by Moscovici/Stanton [16] to higher rank situations. Assume that the twist is acyclic, i.e. $H^*(\Gamma, \chi) = 0$. Recall that the analytic torsion of M with respect to the twist χ is then defined by

$$\tau = \tau_M(\chi) := \sqrt{\prod_{l=1}^n (\det \Delta_{\chi, l})^{(-1)^l l}},$$

where $\Delta_{\chi, l}$ is the Laplacian on l -forms on M twisted with the flat bundle associated to χ . In the following we will omit χ in our notation.

Theorem 5.5 $Z_R(0) = \tau^{-2}$.

Proof: Using Poincare duality, Proposition 5.2 and Proposition 4.7 we obtain

$$\begin{aligned} Z_R(0) &= \prod_{p=0}^{n-1} Z_S\left(\frac{n-1}{2} - p, \sigma^p\right)^{(-1)^p} \\ &= \prod_{p=0}^{n-1} \det(A_M^2(\sigma^p) + \left(\frac{n-1}{2} - p\right)^2)^{(-1)^p} \\ &= \det(A_M(\sigma^+ + \sigma^-))^{(-1)^{(n-1)/2}} \prod_{p=0}^{(n-3)/2} \det(A_M^2(\sigma^p) + \left(\frac{n-1}{2} - p\right)^2)^{2(-1)^p}. \end{aligned}$$

Now $\sigma^p = r(\sum_{l=0}^p (-1)^{l-p} \lambda^l)$ for $p \leq (n-1)/2$ and $\Delta_l = A_M^2(\sigma^l) + (\frac{n-1}{2} - l)^2$. Thus

$$\begin{aligned} Z_R(0) &= \left(\prod_{l=0}^{(n-1)/2} \det(\Delta_l)^{(-1)^l} \right) \prod_{p=0}^{(n-3)/2} \prod_{l=0}^p \det(\Delta_l)^{2(-1)^l} \\ &= \prod_{l=0}^{(n-1)/2} \det(\Delta_l)^{2(\frac{n}{2}-l)(-1)^l} \\ &= \tau^{-2}. \end{aligned}$$

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